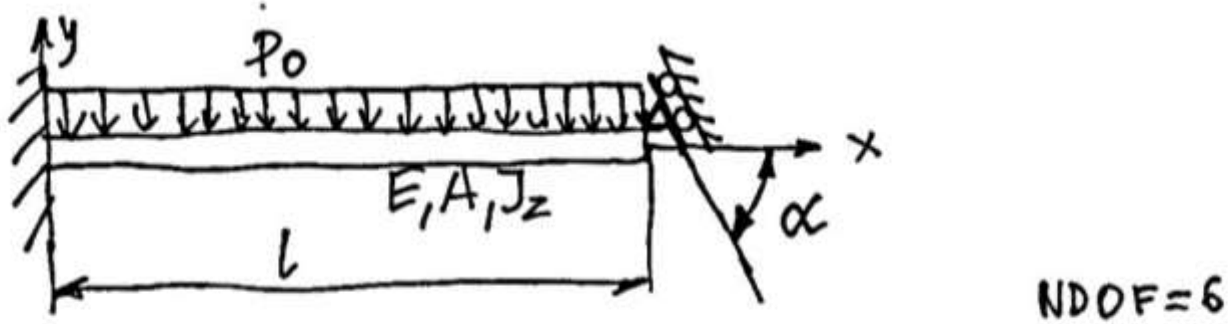
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 10D. Frames - examples

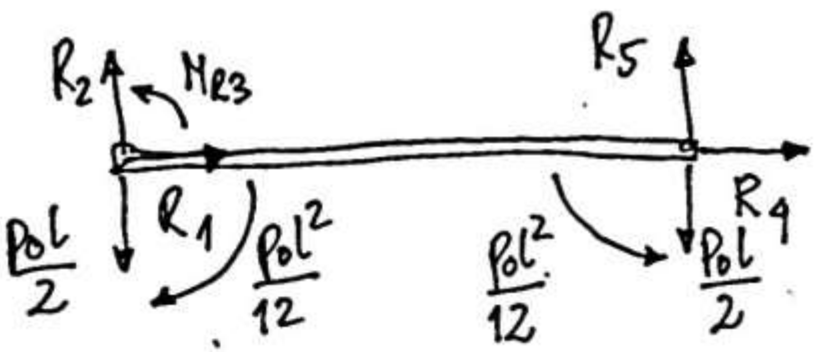
05.2025

Przykład Build a 2D FEM model of the frame. Determine nodal displacements, stresses and reactions.



1°)

$$\underset{6\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$



$$\underset{6\times 1}{\{F\}} = \begin{Bmatrix} R_1 \\ R_2 - \frac{p_0 l}{2} \\ M_{R3} - \frac{p_0 l^2}{12} \\ R_4 \\ R_5 - \frac{p_0 l}{2} \\ \frac{p_0 l^2}{12} \end{Bmatrix}$$

$$[K]_{6\times 6} = \begin{bmatrix} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & d & 0 & -b & d \\ 0 & d & m & 0 & -d & r \\ -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -d & 0 & b & -d \\ 0 & d & r & 0 & -d & m \end{bmatrix}$$

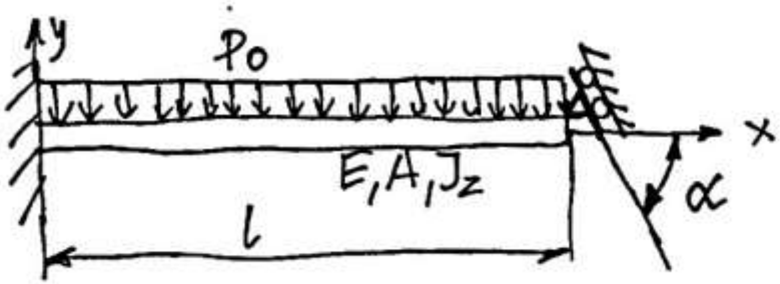
$$a = \frac{EA}{l}$$

$$b = \frac{12EJ_z}{l^3}$$

$$d = \frac{6EJ_z}{l^2}$$

$$m = \frac{4EJ_z}{l}$$

$$r = \frac{2EJ_z}{l}$$



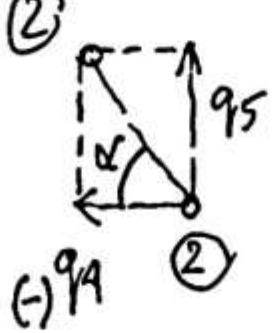
Boundary conditions:

$q_1 = 0$

$q_2 = 0$

$q_3 = 0$

NOF = 4



$$tg\alpha = \frac{q_5}{-q_4} \Rightarrow q_5 = -q_4 \cdot tg\alpha$$

$$N = NDOF - NOF = 6 - 4 = 2$$

Unknown independent degrees of freedom:

$$\underset{NDOF\times 1}{\{q\}} = \underset{NDOF\times N}{[C]} \cdot \underset{N\times 1}{\{q\}} \quad ; \quad \underset{6\times 1}{\{q\}} = \underset{6\times 2}{[C]} \cdot \underset{2\times 1}{\{q\}}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -tg\alpha & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_4 \\ q_6 \end{Bmatrix} \quad ;$$

$$\underset{1\times 6}{\lfloor q \rfloor} = \underset{1\times 2}{\lfloor q \rfloor} \cdot \underset{2\times 6}{[C]^T}$$

$$\underset{2\times 6}{[C]^T} = \begin{bmatrix} 0 & 0 & 0 & 1 & -tg\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$tg\alpha = t$$

Total potential energy:

$$V = \frac{1}{2} \underset{1\times6}{\lfloor q \rfloor} \underset{6\times6}{[K]} \underset{6\times1}{\{q\}} - \underset{1\times6}{\lfloor q \rfloor} \cdot \underset{6\times1}{\{F\}} =$$

$$= \frac{1}{2} \underset{1\times2}{\lfloor q \rfloor} \cdot \underset{2\times6}{[C]^T} \cdot \underset{6\times6}{[K]} \cdot \underset{6\times2}{[C]} \cdot \underset{2\times1}{\{q\}} - \underset{1\times2}{\lfloor q \rfloor} \underset{2\times6}{[C]^T} \cdot \underset{6\times1}{\{F\}} =$$

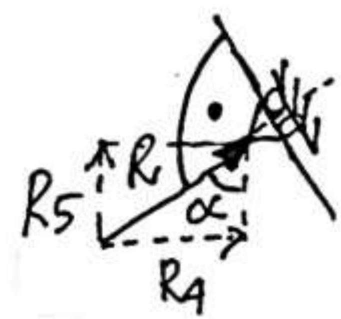
$$= \frac{1}{2} \underset{1\times2}{\lfloor q \rfloor} \underset{2\times2}{[K]} \cdot \underset{2\times1}{\{q\}} - \underset{1\times2}{\lfloor q \rfloor} \cdot \underset{2\times1}{\{F\}} \Rightarrow \underset{2\times2}{[K]} \cdot \underset{2\times1}{\{q\}} = \underset{2\times1}{\{F\}}$$

(boundary conditions taken into account)

$$\underset{6\times6}{[K]}\cdot\underset{6\times2}{[C]}=\begin{bmatrix}-a & 0\\ bt & d\\ td & r\\ a & 0\\ -bt & -d\\ td & m\end{bmatrix}\quad ; \quad \underbrace{\underset{2\times6}{[C]}^{T}\cdot\left(\underset{6\times6}{[K]}\cdot\underset{6\times2}{[C]}\right)}_{\underset{2\times2}{[K]}}=\left[\begin{array}{c|c}a+bt^{2} & td\\ \hline td & m\end{array}\right]$$

$$\underset{2\times6}{[C]}^{T}\cdot\underset{6\times1}{\{F\}}=\begin{Bmatrix}R_4-t\left(R_5-\frac{p_oL}{2}\right)\\ \frac{p_ol^{2}}{12}\end{Bmatrix}=\begin{Bmatrix}\frac{p_oL}{2}\cdot t\\ \frac{p_ol^{2}}{12}\end{Bmatrix}$$

$$\frac{R_4}{R_5}=tg\alpha=t \qquad \boxed{R_4-t\cdot R_5=0}$$



$$\begin{cases} (a+bt^2)\,q_4 + td\cdot q_6 = \frac{P_0 l}{2}\cdot t \\ td\cdot q_4 + m\cdot q_6 = \frac{P_0 l^2}{12} \end{cases}$$

$$\mathrm{II} \rightarrow q_6 = \frac{\frac{P_0 l^2}{12} - t\cdot d\cdot q_4}{m}$$

$$\rightarrow \mathrm{I} \quad (a+bt^2)\,q_4 + \frac{t\cdot d}{m}\left(\frac{P_0 l^2}{12} - t\cdot d\cdot q_4\right) = \frac{P_0 l}{2}\cdot t$$

$$\left(a + bt^2 - \frac{t^2 d^2}{m}\right) q_4 = \frac{P_0 l}{2} t - \frac{P_0 l^2 d}{12 m}\cdot t$$

$$q_4 = \frac{\frac{P_0 l t}{12}\left(6 - \frac{ld}{m}\right)}{a + bt^2 - \frac{d^2}{m} t^2} = \frac{\frac{P_0 l t}{12}\left(6 - \frac{l\cdot 6EJ_z\cdot l}{l^2\cdot 4EJ_z}\right)}{\frac{EA}{l} + \frac{12EJ_z}{l^3} t^2 - \frac{36E^2J_z^2\cdot l}{l^4\cdot 4EJ_z} t^2} =$$

$$= \frac{\frac{3}{8} P_0 l t}{\frac{EA}{l} + \frac{3EJ_z}{l^3} t^2} = \frac{\frac{3}{8t} P_0 l}{\frac{EA}{l t^2} + \frac{3EJ_z}{l^3}}$$

$$q_5 = -q_4 \cdot t = -\frac{\frac{3}{8} p_0 l t^2}{\frac{EA}{l} + \frac{3EJ_z}{l^3} t^2} = \frac{-\frac{3}{8} p_0 l}{\frac{EA}{lt^2} + \frac{3EJ_z}{l^3}}$$

$$\alpha = 0° : \quad q_4 = 0 \ , \quad q_5 = 0$$

$$\alpha = 90° \rightarrow t \rightarrow \infty : \quad q_4 = 0 \ , \quad q_5 = -\frac{3}{24} \frac{p_0 l^4}{EJ_z}$$

$$q_6 = \frac{\frac{p_0 l^2}{12} - t \cdot d \cdot \frac{3}{8t} \cdot p_0 \cdot l \Big/ \left(\frac{EA}{l t^2} + \frac{3EJ_z}{l^3} \right)}{m} =$$

$$= \frac{\left(\frac{p_0 l^2}{12} - \frac{3}{8} p_0 l \cdot \frac{6EJ_z}{l^2 \left(\frac{EA}{l t^2} + \frac{3EJ_z}{l^3} \right)} \right) l}{4EJ_z} =$$

$$= \frac{\frac{p_0 l^3}{12} - \frac{18}{8} p_0 \cdot \frac{1}{\frac{A}{J_z l t^2} + \frac{3}{l^3}}}{4EJ_z} = \frac{\frac{p_0 l^3}{12} - \frac{9}{4} \cdot \frac{p_0 l^3}{\frac{A l^2}{J_z t^2} + 3}}{4EJ_z} =$$

$$= \frac{p_0 l^3}{48EJ_z} \left(1 - \frac{27}{\frac{A l^2}{J_z t^2} + 3} \right)$$

Reactions:

$$\underset{6\times 6}{[K]} \cdot \underset{6\times 1}{\{q\}} = \underset{6\times 1}{\{F\}}$$

$$-a \cdot q_4 = R_1$$

$$-b \cdot q_5 + d \cdot q_6 = R_2 - \frac{p_0 L}{2}$$

$$-d \cdot q_5 + r \cdot q_6 = M_{R3} - \frac{p_0 L^2}{12}$$

$$a \cdot q_4 = R_4$$

$$b \cdot q_5 - d \cdot q_6 = R_5 - \frac{p_0 L}{2}$$

$$R_1 = -\frac{EA}{L} \cdot q_4$$

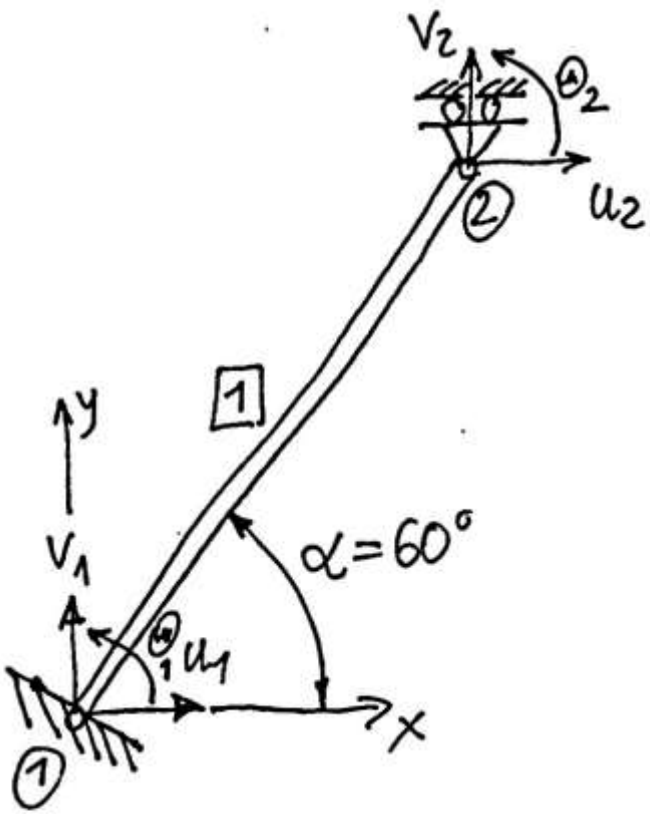
$$R_2 = -\frac{12EJ_z}{L^3} \cdot q_5 + \frac{6EJ_z}{L^2} \cdot q_6 + \frac{p_0 L}{2}$$

$$M_{R3} = -\frac{6EJ_z}{L^2} \cdot q_5 + \frac{2EJ_z}{L} \cdot q_6 + \frac{p_0 L^2}{12}$$

$$R_4 = \frac{EA}{L} \cdot q_4$$

$$R_5 = \frac{12EJ_z}{L^3} q_5 - \frac{6EJ_z}{L^2} \cdot q_6 + \frac{p_0 L}{2}$$

2ª)

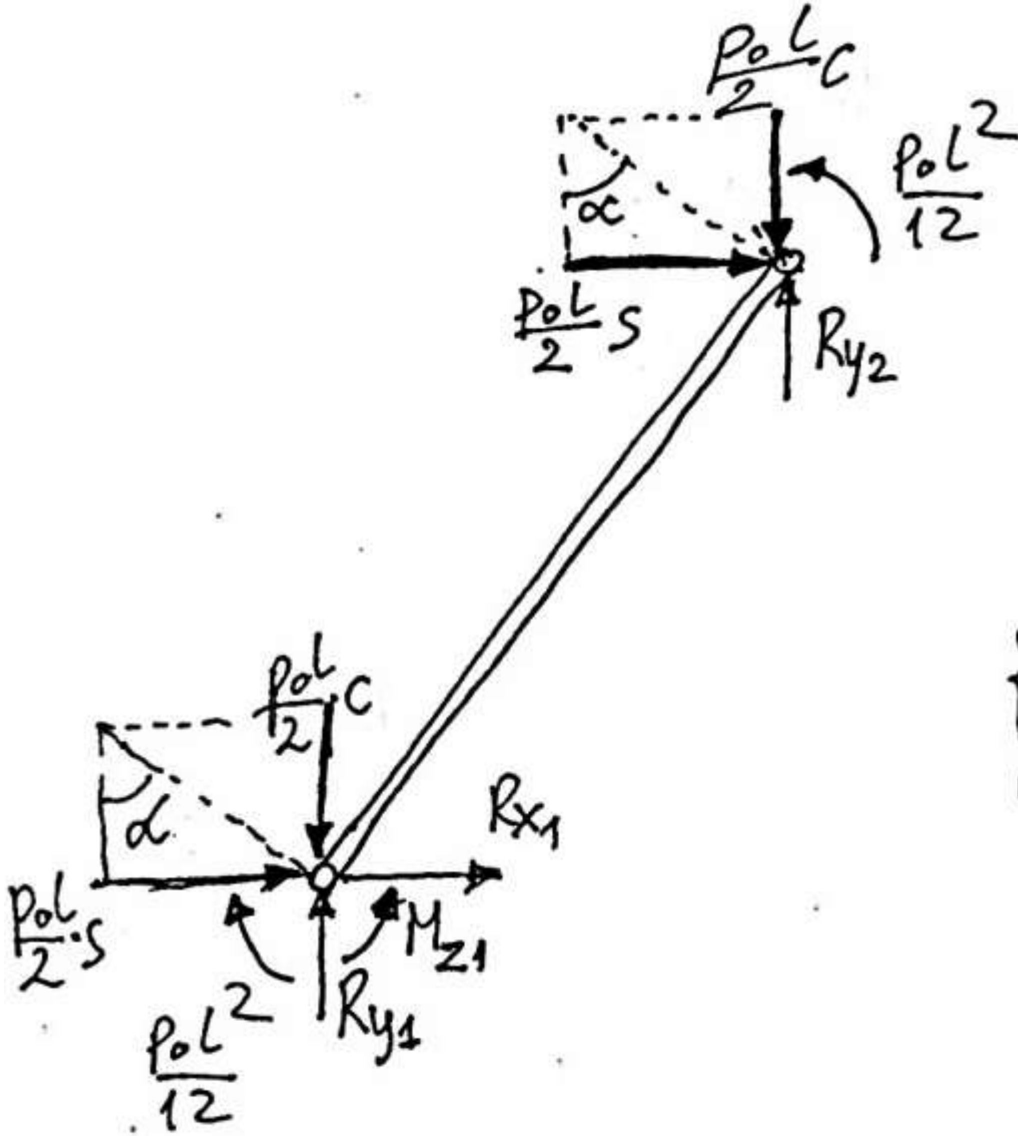


$$c = \frac{1}{2} \qquad t = \frac{s}{c} = \sqrt{3}$$

$$s = \frac{\sqrt{3}}{2}$$

$$\underset{1\times 6}{\lfloor q_g \rfloor_1} = \lfloor u_1, v_1, \theta_1, u_2, v_2, \theta_2 \rfloor$$

$$\underset{1\times 6}{\lfloor q \rfloor} = \underset{1\times 6}{\lfloor q_g \rfloor_1}$$



$$\underset{6\times 1}{\{F\}} = \begin{Bmatrix} R_{x_1} + \frac{p_o L}{2} \cdot s \\ R_{y_1} - \frac{p_o L}{2} \cdot c \\ M_{z_1} - \frac{p_o L^2}{12} \\ \frac{p_o L}{2} \\ R_{y_2} - \frac{p_o L}{2} \cdot c \\ \frac{p_o L^2}{12} \end{Bmatrix}$$

$$\underset{6\times 6}{[K]_1} = \begin{bmatrix} a & 0 & 0 & -a & 0 & 0 \\ 0 & b & d & 0 & -b & d \\ 0 & d & m & 0 & -d & r \\ -a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -d & 0 & b & -d \\ 0 & d & r & 0 & -d & m \end{bmatrix}$$

$$a = \frac{EA}{l}$$

$$b = \frac{12EJ_z}{l^3}$$

$$d = \frac{6EJ_z}{l^2}$$

$$m = \frac{4EJ_z}{l}$$

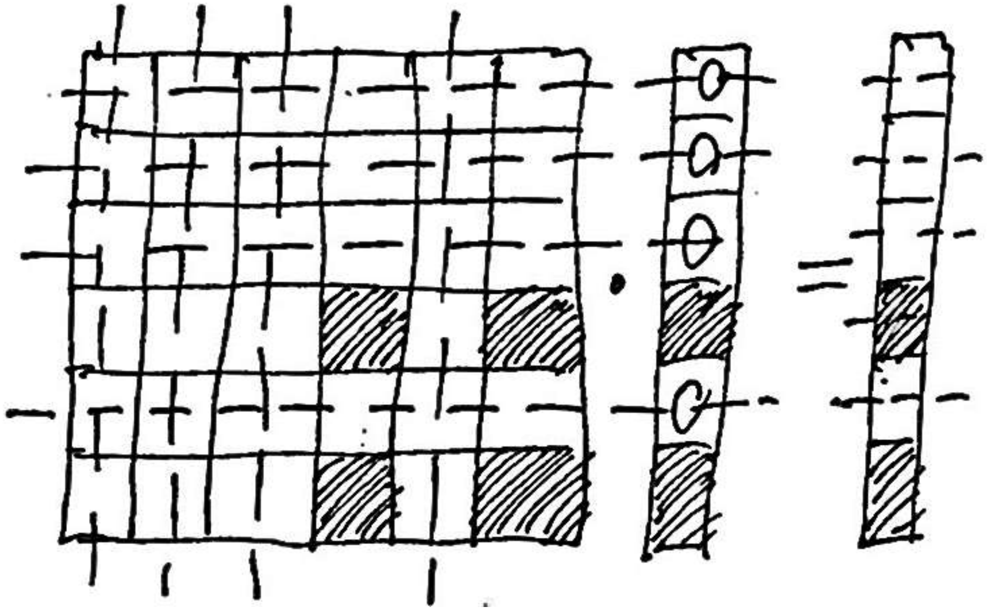
$$r = \frac{2EJ_z}{l}$$

$$\underset{6\times 6}{[T_f]_1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \quad \underset{6\times 6}{[T_f]_1^T} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times 6}{[K]} = \underset{6\times 6}{[k_g]_1} = \underset{6\times 6}{[T_f]_1^T} \cdot \underset{6\times 6}{[k]_1} \cdot \underset{6\times 6}{[T_f]_1}$$

$$\underset{6\times 6}{[K]} \cdot \underset{6\times 1}{\{q\}} = \underset{6\times 1}{\{F\}}$$ + boundary conditions:

$u_1 = 0$, $v_1 = 0$, $\theta_1 = 0$, $V_2 = 0$



$$\begin{bmatrix} k_{44} & k_{46} \\ k_{64} & k_{66} \end{bmatrix} \cdot \begin{Bmatrix} u_2 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} \frac{p_0 l}{2} \\ \frac{p_0 l^2}{12} \end{Bmatrix}$$

$$\rightarrow \quad u_2 = \frac{\frac{3}{8 s_1} p_0 l}{\frac{EA}{l \cdot t^2} + \frac{3EJ_2}{l^3}}$$

$$\varphi_2 = \frac{p_0 l^3}{48 E J_z}\left(1 - \frac{27}{\frac{A l^2}{J_z \cdot t^2} + 3}\right)$$

$$\underset{6\times 1}{\{q\}_1} = \underset{6\times 6}{[T_f]_1} \cdot \underset{6\times 1}{\{q_g\}_1}$$

$$q_1 = 0,\ q_2 = 0,\ q_3 = 0$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = \underset{6\times 6}{[T_f]_1} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_2 \\ 0 \\ \Theta_2 \end{Bmatrix}$$

$$q_4 = \frac{\frac{3}{8t_1} p_0 l}{\frac{EA}{lt^2} + \frac{3EJ_z}{l^3}}$$

$$q_5 = \frac{-\frac{3}{8} p_0 l}{\frac{EA}{lt^2} + \frac{3EJ_z}{l^3}}$$

$$q_6 = \frac{p_0 l^3}{48 EJ_z}\left(1 - \frac{27}{\frac{Al^2}{J_z t^2} + 3}\right)$$

(same as item 1)

Reactions:

$$\underset{6\times6}{[K]} \cdot \underset{6\times1}{\{q\}} = \underset{6\times1}{\{F\}}$$

$$k_{14} \cdot u_2 + k_{16} \cdot \theta_2 = R_{x1} + \frac{p_0 l}{2} \cdot s$$

$$k_{24} \cdot u_2 + k_{26} \cdot \theta_2 = R_{y1} - \frac{p_0 l}{2} \cdot c$$

$$k_{34} \cdot u_2 + k_{36} \theta_2 = M_{z1} - \frac{p_0 l^2}{12}$$

$$k_{54} \cdot u_2 + k_{56} \cdot \theta_2 = R_{y2} - \frac{p_0 l}{2} \cdot c$$

$$R_{x1} = k_{14} \cdot u_2 + k_{16} \theta_2 - \frac{p_0 l}{2} \cdot s$$

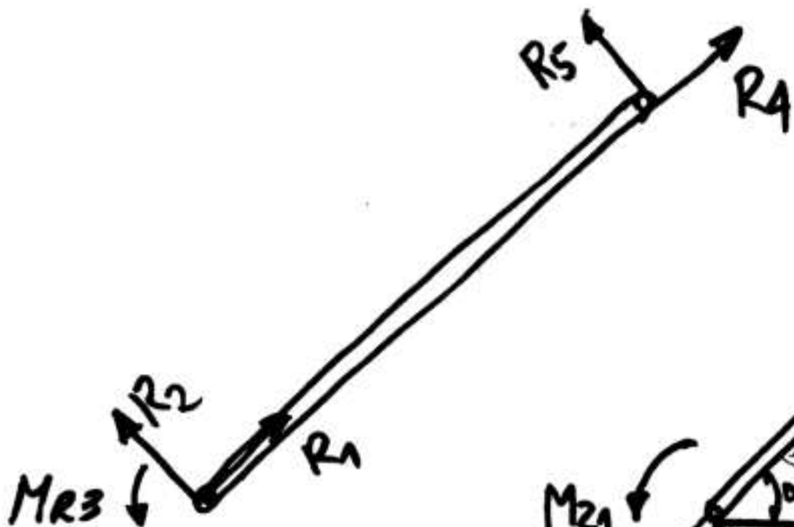
$$R_{y1} = k_{24} \cdot u_2 + k_{26} \theta_2 + \frac{p_0 l}{2} \cdot c$$

$$M_{z1} = k_{34} u_2 + k_{36} \theta_2 + \frac{p_0 l^2}{12}$$

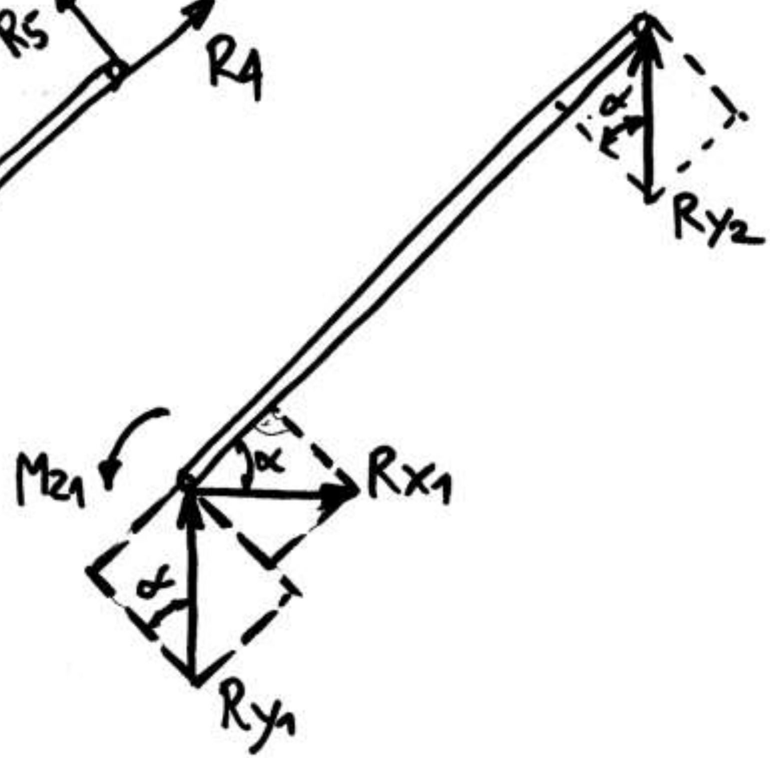
$$R_{y2} = k_{54} u_2 + k_{56} \cdot \theta_2 + \frac{p_0 l}{2} \cdot c$$

Reactions:

1°)



2°)



$$R_{x_1}\cdot\cos\alpha + R_{y_1}\cdot\sin\alpha = R_1$$
$$-R_{x_1}\cdot\sin\alpha + R_{y_1}\cdot\cos\alpha = R_2$$

$$R_{y_2}\cdot\sin\alpha = R_4$$
$$R_{y_2}\cdot\cos\alpha = R_5$$
$$M_{z_1} = M_{R3}$$